JZS-A Volume 22, Issue 2, December 2020

Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)
http://jzs.univsul.edu.iq

On P −Cancellation Modules

Adil Kadir Jabbar[1] and Dana Jamal Saeed[*2]

1Department of Mathematics, College of Science, University of Sulaimani, Sulaimani, Kurdistan Region, Iraq
2Directorate of Education of Sulaimani, Iraq
**Corresponding author's e-mail: danaj41141@gmail.com*

Article info

Original: 20 February 2020
Revised: 5 September 2020
Accepted: 26 September 2020

Abstract

In this paper, the concept of P −cancellation modules is introduced, where P is a prime ideal of a commutative ring R with identity, also the effect of localization on cancellation modules is studied and some properties of this type of modules are given, furthermore some conditions are given which make P −cancellation modules as a cancellation and conversely. Also, the effect of ascending chain conditions and descending chain conditions on the P −cancellation modules are studied and different characterizations of this type of cancellation modules are given.

Key Words:
Multiplicative system, cancellation ideal, cancellation module, and P-cancellation module

1. Introduction

In 1972 Gilmer was the first who introduced the concept of cancellation ideals [5]. After that, D. D. Anderson and D. F. Anderson in 1984, studied the concept of cancellation ideals [2] and they proved so many results concerning this type of cancellation ideals. In 1992, A. G. Naoum and A. S. Mijbass in [10], introduced the concept of cancellation and weakly cancellation modules and they proved many properties of these two kinds of cancellation modules. In the present paper, we present the concept of the P −cancellation module, where P is a prime ideal of a commutative ring R with identity.

Throughout this paper, R is a commutative ring with identity, P is a prime ideal of R and M is a left R −module, unless otherwise stated. A non-empty set $\emptyset \neq S \subseteq R$ is said to be a multiplicative system if $0 \notin S$ and S is closed under multiplication [4]. If S is a multiplicative system in R, then we denote the localization of R at S by R_S (or $S^{-1}R$ [4]), which is $R_S = \{\frac{r}{s} : r \in R, s \in S\}$ [4]. If P is a prime ideal of R, then one can easily get that $R \setminus P$ is a multiplicative system in R, and in this case, we use the symbol R_P to signify the localization of R at $R \setminus P$, so that $R_P = \{\frac{r}{p} : r \in R, p \notin P\}$. If A is an ideal of R, then $A_S = \{\frac{a}{s} : a \in A, s \in S\}$ and if P is a prime ideal of R, then $A_P = \{\frac{a}{p} : a \in A, p \notin P\}$. If N is a submodule of M, then $S_M(N) = \{r \in R : rx \in N$, for some $x \in M \setminus N\}$ [3] and if A is an ideal of R, then $S_R(A) = \{r \in R : ra \in A$, for some $a \notin A\}$ [3]. For a prime ideal P of R and a submodule N of M clearly, we have $S = R \setminus P$ is a multiplicative system in R and $S \cap S_M(N) = \phi$ if and only if $S_M(N) \subseteq P$. For a submodule K of M, $(K : M) = \{r \in R : rM \subseteq K\}$ and $ann(M) = (0 : M) = \{r \in R : rM = 0\}$. An R −module M is called a faithful R −module, if $Ann(M) = (0 : M) = 0$ [13]. An R −module

M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$ [1]. Let M be an $R-$module and $m \in M$. Then m is called a torsion element if there exists $0 \neq r \in R$ such that $rm = 0$. Otherwise, m is called a non-torsion element and if all elements of M are non-torsion elements, then M is called a free torsion module [4]. Let R be a commutative ring with identity. We say R satisfies ascending (descending) chain condition on ideals if any ascending (descending) chain of ideals in R is stationary, that is if $I_1 \subseteq I_2 \subseteq \cdots I_n \subseteq \cdots$ $(I_1 \supseteq I_2 \supseteq \cdots I_n \supseteq \cdots)$ is any ascending (descending) chain of ideals in R, then there exists $k \in Z^+$ such that $I_n = I_k$ for all $k \geq n$ and R is called a Noetherian (Artinian) ring if it satisfies the ascending (descending) chain condition for ideals [4]. Let R be a ring, then it is called a semi-local ring if it has a finite number of maximal ideals and it is called a local ring if it has a unique maximal ideal [4]. An $R-$module M is said to be a cancellation (weak cancellation) module whenever A, B are ideals of R and $AM = BM$ then $A = B(A + ann(M) = B + ann(M))$ [11]. Finally, By $J(M)$ we mean the Jacobson radical of a module M while $J(M) = \bigcap_K K$ where K is a maximal submodule of M) and if R is considered as an $R-$module then the Jacobson radical of R, denoted by $J(R)$, is defined by $J(R) = \bigcap_P P$ Where P is a maximal ideal of R [4].

2. Some Properties of $P-$Cancellation Modules

The aim of this section is to introduce the concept of $P-$cancellation modules and demonstrate some basic properties of this concept.

Definition 2.1. Let M be an $R-$module and P be a prime ideal of R. An $R-$module M is called a $P-$cancellation module if M_P is a cancellation module, that is, if $\overline{A}$ and $\overline{B}$ are any ideals of R_P such that $\overline{A}M_P = \overline{B}M_P$, then $\overline{A} = \overline{B}$.

Examples 2.2.
(1) Z is a $P-$cancellation $Z-$module for each prime ideal P of Z.
(2) Q is not a $2Z-$cancellation $Z-$module.

Below we give a result which proves that an ideal of R is a cancellation ideal if and only if its localization is a cancellation ideal at a given prime ideal of R.

Proposition 2.3. Let P be a prime ideal of R such that $S_R(I) \subseteq P$, for every ideal I of R and A be an ideal of R, then A is a cancellation ideal of R if and only if A_P is a cancellation ideal of R_P.

Proof. To show A_P is a cancellation ideal of R_P. Let $\bar{B}A_P = \bar{C}A_P$, where $\bar{B}$ and $\bar{C}$ are two ideals of R_P. Then, $\bar{B} = B_P$ and $\bar{C} = C_P$, for some ideals B and C of R. Then we get, $(BA)_P = B_P A_P = \bar{B}A_P = \bar{C}A_P = C_P A_P = (CA)_P$.To show $BA = CA$. By hypothesis, $S_R(I) \subseteq P$. Now, let $x \in BA$, then $\frac{x}{1} \in (BA)_P = (CA)_P$, then there exists $u \notin P$ such that $ux \in CA$. Now if possible, suppose that $x \notin CA$ which implies that $u \in S_M(CA)$ which is contradiction (since $S_R(CA) \subseteq P$), thus $x \in CA$ and hence, we get $BA \subseteq CA$. By the same technique as in the above we can show that $CA \subseteq BA$ and so we get $BA = CA$ and since A is a cancellation ideal of R, we get $B = C$ and thus $\bar{B} = B_P = C_P = \bar{C}$. Hence A_P is a cancellation ideal of R_P.
Conversely, suppose that A_P is a cancellation ideal of R_P and to show that A is a cancellation ideal of R. Let B, C are ideals of R such that $BA = CA$, then $B_P A_P = (BA)_P = (CA)_P = C_P A_P$ and as A_P is a cancellation ideal of R_P, we get $B_P = C_P$ and since $S_R(B) \subseteq P$ and $S_R(C) \subseteq P$, one can easily get that $B = C$. Hence, A is a cancellation ideal of R.

It is known that, if an $R-$module M is a cancellation module then $M_P \neq 0$, for every maximal ideal P of R **[11, Proposition 2.1]**. Now, we can extend this result to prime ideals for $P-$cancellation modules.

Lemma 2.4. If M is a $P-$cancellation module, where P is a prime ideal of R, then $M_P \neq 0$. In particular, if P is a maximal ideal of R, then $M_P \neq 0$.

Proof. As M is a $P-$cancellation module, we have M_P is a cancellation module and since P_P is a maximal (actually, the unique maximal) ideal of R_P, so by [11, Proposition 2.1], we obtain $(M_P)_{P_P} \neq 0$, this gives that $M_P \neq 0$ (since if $M_P = 0$, then $(M_P)_{P_P} = 0$ which is a contradiction). The proof of the second part follows directly from the certainty that any maximal ideal of a commutative ring with identity is a prime ideal.

In the following result, a condition is given, under which a $P-$cancellation module is a faithful module.

Theorem 2.5. An $R-$module M is faithful if it is $P-$cancellation and $S_M(0) \subseteq P$, where P is prime Ideal.

Proof. To show M is faithful, it is enough to show that $Ann(M) = 0$, so let $r \in Ann(M)$, then $rM = 0$, this implies that $< r >_P M_P = (< r > M)_P \subseteq (rM)_P = 0_P = 0_P M_P$. As M is a $P-$cancellation module, we get that $< r >_P = 0$, and since $\frac{r}{1} \in < r >_P$, so that $\frac{r}{1} = 0$, this implies that $qr = 0$, for some $q \notin P$. Now, if $r \neq 0$, then $q \in S_M(0) \subseteq P$, which makes a contradiction, that is $r = 0$ and hence, $Ann(M) = 0$. Hence, M is a faithful module.

Next, we give two equivalent conditions each makes a given submodule of a $P-$cancellation multiplication $R-$module as a $P-$cancellation submodule.

Proposition 2.6. Let M be a $P-$cancellation $R-$module, where P is a prime ideal of R. If M is a multiplication $R-$module and N is a submodule of M, then the following conditions are equivalent.

(1) N is a $P-$cancellation submodule.

(2) $(N : M)$ is a $P-$cancellation ideal of R.

(3) $N = AM$, where A is an ideal of R satisfies $P-$cancellation property.

Proof. $(1) \Rightarrow (2)$ Let N be submodule of M satisfies $P-$cancellation property. To show $(N : M)$ is a $P-$cancellation ideal of R, it is enough to show that $(N : M)_P$ is a cancellation ideal of R_P, so let $\overline{A}$ and $\overline{B}$ be two ideals of R_P, such that $\overline{A}(N : M)_P = \overline{B}(N : M)_P$, then there exist ideals A and B of R such that $\overline{A} = A_P$ and $\overline{B} = B_P$. As, N is a multiplication submodule, so that $N = (N : M)M$, then $\overline{A}N_P = A_P N_P = \overline{A}((N : M)M)_P = \overline{A}(N : M)_P M_P = \overline{B}(N : M)_P M_P = \overline{B}((N : M)M)_P = \overline{B}N_P$, and as N is a $P-$cancellation module, we have N_P is a cancellation module, so we get $\overline{A} = \overline{B}$. Hence, we obtain $(N : M)_P$ is a cancellation ideal of R_P, that is, $(N : M)$ is a $P-$cancellation ideal of R.

$(2) \Rightarrow (3)$ It is clear that $N = (N : M)M$ and $(N : M)$ is a $P-$cancellation ideal of R, so by putting $A = (N : M)$ we get the result.

$(3) \Rightarrow (1)$ Suppose that, $N = AM$, for the $P-$cancellation ideal A of R and to show that N is a $P-$cancellation submodule. Now, let $\overline{A}$ and $\overline{B}$ be two ideals of R_P such that $\overline{B}N_P = \overline{C}N_P$. Now, $\overline{B}A_P M_P = A_P\overline{B}M_P = A_P\overline{C}M_P = \overline{C}A_P M_P$. Since, M is a $P-$cancellation module, so we get $\overline{B}A_P = \overline{C}A_P$ and since A is a $P-$cancellation ideal, we get $\overline{B} = \overline{C}$. Hence, N is a $P-$cancellation submodule.

Next, we prove some results concerning the effect of ascending chain conditions and descending chain conditions on $P-$cancellation modules and the rings on which the modules are defined.

Proposition 2.7. Let R be a semi-local ring with the maximal ideals $P_1, P_2, \dots, P_k$ and M be a Noetherian P_i-cancellation $R-$module, for every i $(1 \leq i \leq k)$. Then R is a Noetherian ring.

Proof. Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ be any ascending chain of ideals in R, then for any i, we have clearly $(A_1)_{P_i} \subseteq (A_2)_{P_i} \subseteq \cdots \subseteq (A_n)_{P_i} \subseteq \cdots$ is an ascending chain of ideals in R_{P_i} and then for each i, we get $(A_1)_{P_i}M_{P_i} \subseteq (A_2)_{P_i}M_{P_i} \subseteq \cdots \subseteq (A_n)_{P_i}M_{P_i} \subseteq \cdots$ is an ascending chain of submodules in M_{P_i} and as M is Noetherian, we get M_{P_i} is also Noetherian for each i, so that for all i $(1 \leq i \leq k)$, there exists a positive integer $m_i \in N$ such that $(A_{m_i})_{P_i}M_{P_i} = (A_{m_i+1})_{P_i}M_{P_i} = \cdots$. Now, let $m = \max\{m_i\}$, then clearly we have $(A_m)_{P_i}M_{P_i} = (A_{m+1})_{P_i}M_{P_i} = \cdots$ for each i $(1 \leq i \leq k)$ and as each M_i is a P_i-cancellation module, we get

$(A_m)_{P_i} = (A_{m+1})_{P_i} = \cdots$, that means $(A_m)_{P_i} = (A_{m+1})_{P_i} = \cdots$, for each maximal ideal P_i of R, so that we get $A_m = A_{m+1} = \cdots$. Hence, R is a Noetherian ring.

As a corollary to **Proposition 2.7**, we give the following result.

Corollary 2.8. If R is a local ring and M is a Noetherian $P-$cancellation $R-$module, then R is a Noetherian ring.

Proposition 2.9. Let R be a semi-local ring with the maximal ideals $P_1, P_2, \dots, P_k$ and M be an Artinian P_i-cancellation $R-$module for every i $(1 \leq i \leq k)$. Then R is an Artinian ring.

Proof. Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be any descending chain of ideals in R, then for each i, we have clearly $(A_1)_{P_i} \supseteq (A_2)_{P_i} \supseteq \cdots \supseteq (A_n)_{P_i} \supseteq \cdots$ is a descending chain of ideals in R_{P_i} and then for each i, we get $(A_1)_{P_i}M_{P_i} \supseteq (A_2)_{P_i}M_{P_i} \supseteq \cdots \supseteq (A_n)_{P_i}M_{P_i} \supseteq \cdots$ is a descending chain of submodules in M_{P_i} and as M is Artinian, we get M_{P_i} is also Artinian for each i, so that for all i $(1 \leq i \leq k)$, there exists a positive integer $m_i \in N$ such that $(A_{m_i})_{P_i}M_{P_i} = (A_{m_i+1})_{P_i}M_{P_i} = \cdots$. Now, let $m = \max\{m_i\}$, then clearly we have $(A_m)_{P_i}M_{P_i} = (A_{m+1})_{P_i}M_{P_i} = \cdots$ for all i $(1 \leq i \leq k)$ and as each M_i is a P_i-cancellation module, so for all i $(1 \leq i \leq k)$ we get $(A_m)_{P_i} = (A_{m+1})_{P_i} = \cdots$, that means, for each maximal ideal P_i of R, we get $(A_m)_{P_i} = (A_{m+1})_{P_i} = \cdots$, so that we get $A_m = A_{m+1} = \cdots$. Hence, R is an Artinian ring.

As a corollary to **Proposition 2.9**, we give the following corollary.

Corollary 2.10. If R is a local ring and M is an Artinian $P-$cancellation $R-$module. Then R is an Artinian ring.

Now, we prove that the modules which are $P-$cancellation for each prime ideal P are cancellation modules.

Proposition 2.11. Let M be a $P-$cancellation module for all prime ideal P of R, then M is a cancellation module.

Proof. Let P be any maximal ideal of R, then clearly P is a prime ideal of R, so by the given condition, M is a $P-$cancellation module, that is M_P is a cancellation module, and to show M is a cancellation module, let A and B be two ideals of R such that $AM = BM$, then we have $A_PM_P = (AM)_P = (BM)_P = B_PM_P$, and as M_P is a cancellation module, we get $A_P = B_P$ and this last result is true for all maximal ideal P of R, that is by [9, Proposition 3.13], we obtain $A = B$, and hence M is a cancellation module.

Proposition 2.12. If M is a non-zero $P-$cancellation $R-$module such that $S_M(0) \subseteq P$, $S_R(J(R)) \subseteq P$ and $S_R(J(M):M) \subseteq P$, where P is a prime ideal of R, then $(J(M):M) = J(R)$.

Proof. Let P be any maximal ideal of R. If $M_P = 0$, then for any $m \in M$, we have $\frac{m}{1} = 0$, then $pm = 0$, for some $p \notin P$, if $m \neq 0$, then $p \in S_M(0) \subseteq P$, which is a contradiction. Hence, $m = 0$, this gives $M = 0$, that is a contradiction, so that $M_P \neq 0$. As, M is a $P-$cancellation $R-$module, we get M_P is a non-zero cancellation module, so by [12, Theorem 1.8], we get $(J(M_P):M_P) = J(R_P)$. On the other hand, we have $(J(M):M)_P = (J(M)_P:M_P) = (J(M_P):M_P) = J(R_P) = (J(R))_P$, as $S_R(J(R)) \subseteq P$ and $S_R(J(M):M) \subseteq P$, we get $(J(M):M) = J(R)$.

3. Some Characterizations and Properties of $P-$Cancellation Modules

This section is devoted to discussing some characterizations and Properties of $P-$cancellation modules. Initially, we prove that a multiplication module that contains a $P-$cancellation submodule is a $P-$cancellation module and finitely generated.

Theorem 3.1. If M is a multiplication $R-$module which contains a $P-$cancellation submodule N, then M is a finitely generated $P-$cancellation module. Whenever P is any prime ideal of R.

Proof. Since N is a submodule of M also as M is a multiplication $R-$module, then there exists an ideal I of R such that $N = IM$, then $N_P = (IM)_P = I_PM_P$. To show M is a $P-$cancellation module, let $\overline{A}$ and $\overline{B}$ be two ideals of R_P such that $\overline{A}M_P = \overline{B}M_P$, then there exist ideals A, B of R such that $\overline{A} = A_P$ and $\overline{B} = B_P$, so that

$A_PM_P = B_PM_P$. Now, $\overline{A}N_P = A_PN_P = A_PI_PM_P = I_PA_PM_P = I_PB_PM_P = B_PI_PM_P = \overline{B}N_P$ and as N is a P −cancellation submodule, we have $\overline{A} = \overline{B}$, so that M is P −cancellation submodule. The proof that M is finitely generated follows directly from [10, Corollary 3.11].

Now, we prove that an R −module M is P −cancellation if and only if its localization is P_P −cancellation, where P is a prime ideal of R.

Proposition 3.2. For a prime ideal P of R, an R −module M is a P −cancellation module if and only if M_P is P_P −cancellation.

Proof. Suppose that M is a P −cancellation module. So that M_P is a cancellation module. To show M_P is a P_P −cancellation module, it is enough to show that $(M_P)_{P_P}$ is a cancellation module. Note that P_P is the unique maximal ideal of R_P and hence it is a prime ideal of R_P. Now, let H and K be two ideals of $(R_P)_{P_P}$ such that $H(M_P)_{P_P} = K(M_P)_{P_P}$, then there exist two ideals $\overline{A}$ and $\overline{B}$ of R_P such that $H = (\overline{A})_{P_P}$ and $K = (\overline{B})_{P_P}$, this gives that $(\overline{A})_{P_P}(M_P)_{P_P} = (\overline{B})_{P_P}(M_P)_{P_P}$, then $(\overline{A}M_P)_{P_P} = (\overline{B}M_P)_{P_P}$ and as P_P is the unique maximal ideal of R_P. That is by [7, Corollary 2.2], we obtain $\overline{A}M_P = \overline{B}M_P$, and as M_P is a cancellation module, we get $\overline{A} = \overline{B}$, so that $H = (\overline{A})_{P_P} = (\overline{B})_{P_P} = K$. Hence, M_P is a P_P −cancellation module.

Conversely, suppose that M_P is a P_P −cancellation module. To show M is a P −cancellation module, it is enough to show that M_P is a cancellation module, so let $\overline{A}$ and $\overline{B}$ be two ideals of R_P such that $\overline{A}M_P = \overline{B}M_P$. Then, we have $(\overline{A})_{P_P}(M_P)_{P_P} = (\overline{A}M_P)_{P_P} = (\overline{B}M_P)_{P_P} = (\overline{B})_{P_P}(M_P)_{P_P}$ and since, M_P is a P_P −cancellation module, so we get $(M_P)_{P_P}$ is a cancellation module, so that we get $(\overline{A})_{P_P} = (\overline{B})_{P_P}$, and since, P_P is the unique maximal ideal of R_P, that is by [7, Corollary 2.2], we have $\overline{A} = \overline{B}$, which implies that M_P is a cancellation module, that is M is a P −cancellation module.

Next, we determine some characterizations of P −cancellation modules.

Theorem 3.3. Let M be an R −module and P be a prime ideal of R such that $S_M(N), S_R(I) \subseteq P$ for every ideal I and every submodule, then the following are equivalent.

(1) M is a P −cancellation module.

(2) If $AM \subseteq BM$ where A and B are ideals of R, then $A \subseteq B$.

(3) If $(a)M \subseteq BM$ where $a \in R$ and B is an ideal of R, then $a \in B$.

(4) $(AM : M) = A$, for all ideal A of R.

(5) $(AM : BM) = (A : B)$, for all ideals A and B of R.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Suppose that M is a P −cancellation module and A, B are two ideals of R such that $AM \subseteq BM$ and we have to show that $A \subseteq B$. We have, $A_PM_P = (AM)_P \subseteq (BM)_P = B_PM_P$ and as M_P is a cancellation module, so by [10, Theorem 1.9], we get $A_P \subseteq B_P$ and as $S_R(B) \subseteq P$, we get $A \subseteq B$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ Suppose that (2) holds and let $(a)M \subseteq BM$ so by (**2**) $(a) \subseteq B$ that is $a \in B$.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$ Suppose that (3) holds and let A be an ideal of R. To show $(AM : M) = A$, so let $x \in (AM : M)$ then $xM \subseteq AM$, this implies that $xM \subseteq (x)M \subseteq AM$ so by (**3**) we get $x \in A$ hence $(AM : M) \subseteq A$. Next let $x \in A$, then we get $xM \subseteq AM$, this gives $x \in (AM : M)$, so that $A \subseteq (AM : M)$, thus $(AM : M) = A$.

$(\mathbf{4}) \Rightarrow (\mathbf{5})$ Suppose that (4) holds and let A, B are ideals of R and to show $(AM : BM) = (A : B)$. Let $x \in (AM : BM)$ then $x \in \big((AM : M) : B\big)$ but by (**4**) we have $(AM : M) = A$, so that $x \in (A : B)$ and thus $(AM : BM) \subseteq (A : B)$. Next, let $x \in (A : B)$ so by (4) we have $(AM : M) = A$ that is $x \in (A : B) = \big((AM : M) : B\big) = (AM : BM)$ by [9, Proposition 2.3], so that we get $(A : B) \subseteq (AM : BM)$. Hence, $(AM : BM) = (A : B)$.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$. Suppose that the condition (**5**) is satisfied. To show M is a P −cancellation module it is enough to show that M_P is a cancellation module. Let $\bar{A}$ and $\bar{B}$ be ideals of R_P, then there exist ideals A and B of R such that $\bar{A} = A_P$ and $\bar{B} = B_P$. By (**5**), we have $(AM : BM) = (A : B)$. As $S_M(AM) \subseteq P$, we get $(\bar{A}M_P : \bar{B}M_P) =$

$(A_P M_P : B_P M_P) = (AM : BM)_P = (A : B)_P = (A_P : B_P) = (\bar{A} : \bar{B})$, so by [10, Theorem 1.10], we get M_P is a cancellation module and hence, M is a P −cancellation module.

The following result shows that an R −module M is a P −cancellation module whenever its homomorphic image is a P −cancellation module.

Proposition 3.4. Let P be a prime ideal of R and M, N be two R −modules with N as a homomorphic image of M which is a P −cancellation module, then M is also a P −cancellation R −module.

Proof. Since N is a homomorphic image of M, so let $f : M \to N$ be an epimorphism, so that $f(M) = N$. We will show that M is a P −cancellation module. Let $\bar{A}$ and $\bar{B}$ be two ideals of R_P such that $\bar{A}M_P = \bar{B}M_P$, then there exist ideals A and B of R such that $\bar{A} = A_P$ and $\bar{B} = B_P$. Let, f_P be the induced R_P −morphism by f, then we get $f_P(M_P) = (f(M))_P = N_P$. We have, $\bar{A}N_P = A_P f_p(M_P) = f_P(A_P M_P) = f_P(\bar{A}M_P) = f_P(\bar{B}M_P) = f_P(B_P M_P) = B_P f_P(M_P) = \bar{B}N_P$ and as N is a P −cancellation R −module, we get $\bar{A} = \bar{B}$. Hence, M is a P −cancellation R −module.

The converse of the above Proposition, need not be true in general as it is shown by the following example.

Example 3.5. Consider $M = Q \oplus Z$ as a Z −module. Take the prime ideal $P = 2Z$, of Z. As Z is a Z −module, by Example 2.2 (1), we have Z is a $2Z$ −cancellation Z −module and as Z is a direct summand of M, so by Proposition 3.4, we get M is a $2Z$ −cancellation Z −module. On the other hand, we have Q is a homomorphic image of M which is not a $2Z$ −cancellation Z −module (see Example 2.2 (2)).

Next, we prove the following result.

Proposition 3.6. Let M be an R −module which has a P −cancellation direct summand L, then M is also a P −cancellation module, where P is a prime ideal of R.

Proof. As L is a direct summand of M, so let $M = L \oplus N$, where N is a submodule of M. Next, let $\pi : M \to L$ defined by $\pi(l + n) = l$, for all $l \in L$ and $n \in N$, be the natural projection, then clearly $\pi(M) = L$, that means, L is a homomorphic image of M, so by Proposition 3.4, M is a P −cancellation module.

In the next two results, we prove that under certain conditions the summation and the intersection of two P −cancellation submodules of an R −module are also P −cancellation submodules.

Proposition 3.7. If M is an R −module and L, N are two P −cancellation submodules of M such that $S_M(L) \subseteq P$, $S_M(N) \subseteq P$ and $R = (N : L) + (L : N)$, then $N + L$ is also a P −cancellation submodule. Where P is a prime ideal of R.

Proof. As L, N are P −cancellation submodules, we have L_P and N_P are cancellation submodules of M and since, $R = (N : L) + (L : N)$, so we get $R_P = ((N : L) + (L : N))_P = (N : L)_P + (L : N)_P = (N_P : L_P) + (L_P : N_P)$. Then, by [8, page 116], we get $N_P + L_P$ is a cancellation module and since we have, $(N + L)_P = N_P + L_P$, so that $(N + L)_P$ is a cancellation module, that is $N + L$ is a P −cancellation module.

Proposition 3.8. If M is an R −module and L, N are two P −cancellation submodules of M such that $S_M(L) \subseteq P$, $S_M(N) \subseteq P$ and $R = (N : L) + (L : N)$, then $N \cap L$ is also a P −cancellation module. Where P be a prime ideal of R.

Proof. As L, N are P −cancellation submodules, we have L_P and N_P are cancellation submodules of M and since, $R = (N : L) + (L : N)$, so we get $R_P = ((N : L) + (L : N))_P = (N : L)_P + (L : N)_P = (N_P : L_P) + (L_P : N_P)$. Then, by [8, page 116], we get $N_P \cap L_P$ is a cancellation module and since we have, $(N \cap L)_P = N_P \cap L_P$, so that $(N \cap L)_P$ is a cancellation module, that is $N \cap L$ is a P −cancellation module.

The next result shows that under certain conditions nonzero cyclic modules are P −cancellation.

Proposition 3.9. Let $M = Rm$, where $0 \neq m \in M$ and P be prime ideal of R, then M is a P −cancellation module if $S_M(0) = 0$.

Proof. We have $M = Rm$, for $m \in M$ and to show M is a P −cancellation module, let $\overline{A}$ and $\overline{B}$ be any two ideals of R_P such that $\overline{A}M_P = \overline{B}M_P$, then there exist ideals A and B of R such that $\overline{A} = A_P$ and $\overline{B} = B_P$, then we get $A_P(Rm)_P = B_P(Rm)_P$. Now let $\frac{a}{s} \in A_P$, where $a \in A$ and $s \notin P$. As $\frac{m}{1} \in (Rm)_P$, we get $\frac{am}{s} = \frac{a}{s}\frac{m}{1} \in A_P(Rm)_P = B_P(Rm)_P = (BRm)_P$, then there exists $u \notin P$ such that $uam \in BRm$, then $uam = \sum_{i=1}^{n} b_i r_i m = (\sum_{i=1}^{n} b_i r_i)m$, where $b_i \in B$ and $r_i \in R$, for $i(1 \leq i \leq n)$, this implies that $(ua - \sum_{i=1}^{n} b_i r_i)m = 0$. Now, As $0 \neq m$, so $ua - \sum_{i=1}^{n} b_i r_i \in S_M(0) = 0$ and thus we get $ua - \sum_{i=1}^{n} b_i r_i = 0$, that is $ua = \sum_{i=1}^{n} b_i r_i \in B$, then $\frac{a}{s} = \frac{u}{u}\frac{a}{s} = \frac{ua}{us} \in B_S$, so that $A_P \subseteq B_P$. By using the same technique, we get $B_P \subseteq A_P$, so $A_P = B_P$, that is, $\overline{A} = \overline{B}$, so M is a P −cancellation module.

Next, we prove that cyclic R −modules which are generated by non-torsion elements are P −cancellation modules.

Proposition 3.10. Let P be a prime ideal of R, then every cyclic R −module which is generated by a non-torsion element is a P −cancellation module.

Proof. Suppose that $M = Rm$, where m is a non-torsion element of M and to show that M is a P −cancellation module. Let $\overline{A}$ and $\overline{B}$ be any ideals of R_P such that $\overline{A}M_P = \overline{B}M_P$, then there exist ideals A and B of R such that $\overline{A} = A_P$ and $\overline{B} = B_P$, then we get $A_P(Rm)_P = B_P(Rm)_P$. Now let $\frac{a}{s} \in A_P$, where $a \in R$ and $s \notin P$, then $aq \in A$, for some $q \notin P$. As $\frac{m}{1} \in (Rm)_P$, we get $\frac{qam}{qs} = \frac{a}{s}\frac{m}{1} \in A_P(Rm)_P = B_P(Rm)_P = (BRm)_P$, then there exists $u \notin P$ such that $uqam \in BRm$, then $uqam = \sum_{i=1}^{n} b_i r_i m = (\sum_{i=1}^{n} b_i r_i)m$, where $b_i \in B$ and $r_i \in R$, for $i(1 \leq i \leq n)$, then $(uqa - \sum_{i=1}^{n} b_i r_i)m = 0$. As m is a non-torsion element, we get $uqa - \sum_{i=1}^{n} b_i r_i = 0$, that is $uqa = \sum_{i=1}^{n} b_i r_i \in B$, then $\frac{a}{s} = \frac{u}{u}\frac{q}{q}\frac{a}{s} = \frac{uqa}{uqs} \in B_S$, so that $A_P \subseteq B_P$. By using the same technique, we get $B_P \subseteq A_P$, so $A_P = B_P$, that is, $\overline{A} = \overline{B}$, so that M is a P −cancellation module.

Finally, we take an example that illustrates that, if the generator of cyclic R −modules M is a torsion element the M may not be the P −cancellation module.

Example 3.11. Z_2 as a Z −module isn't a $< 3 >$ −cancellation module. We have $Z_2 = \langle \bar{1} \rangle$ and $\langle 3 \rangle$ is a prime ideal of Z and clearly $\langle 2 \rangle_{\langle 3 \rangle} Z_{2\langle 3 \rangle} = \langle 0 \rangle_{\langle 3 \rangle} = \langle 0 \rangle_{\langle 3 \rangle} Z_{2\langle 3 \rangle}$ but $\langle 0 \rangle_{\langle 3 \rangle} \neq \langle 2 \rangle_{\langle 3 \rangle}$, so that Z_2 is not a $< 3 >$ −cancellation module.

4. Cancellation and P −Cancellation Modules

In the last section, we focus on the relations that combining cancellation modules and P −cancellation modules. So, the following two results give us a relation that combines cancellation (P −cancellation) ideals of a ring and cancellation (P −cancellation) submodules in P −cancellation modules.

Proposition 4.1. Let M be P −cancellation, P be a prime ideal with $S_M(K), S_R(I) \subseteq P$, for every submodule K and every ideal I, then an ideal A is cancellation if and only if AM is cancellation.

Proof. (⇒)Assume that AM is a cancellation module and to show A is a cancellation ideal of R, so let $BA = CA$ where B and C are ideals of R. Now clearly, we have $BAM = CAM$ then as AM is a cancellation module so $B = C$ and hence A is a cancellation ideal of R.

(⇐) Let A be a cancellation ideal of R, since $S_R(I) \subseteq P$ for every ideal I of R so by Proposition 2.3, we get A_P is a cancellation ideal of R_P and as M_P is a cancellation module by [10, Proposition 1.5], we have $(AM)_P$ is a cancellation module and since, $(AM)_P = A_P M_P$, so that $A_P M_P$ is a cancellation module. To show AM is a cancellation module, so let $BAM = CAM$, where B and C are ideals of R, then we get B_P and C_P are ideals of R_P . Now, we have $B_P A_P M_P = (BAM)_P = (CAM)_P = C_P A_P M_P$ and as $A_P M_P$ is a cancellation module, so we get $B_P = C_P$. As $S_R(B) \subseteq P$ and $S_R(C) \subseteq P$, we get $B = C$. Hence AM is a cancellation module.

Theorem 4.2. Let M be a P −cancellation R −module, P is a prime ideal of R. If A is an ideal of R, then AM is a P −cancellation submodule if and only if A is a P −cancellation ideal of R.

Proof. (⇒) Suppose that AM is a P −cancellation module, where P is a prime ideal of R, to show A is a P −cancellation ideal of R, let $\overline{B}$ and $\overline{C}$ be any ideals of R_P such that $\overline{B}A_P = \overline{C}A_P$. Then, there exist two ideals B and C of R such that $\overline{B} = B_P$ and $\overline{C} = C_P$, so that $B_PA_P = C_PA_P$. Then, clearly, we have $B_PA_PM_P = C_PA_PM_P$, so that $B_P(AM)_P = B_P(A_PM_P) = C_P(A_PM_P) = C_P(AM)_P$ and since AM is a P −cancellation module, we get $B_P = C_P$, that is $\overline{B} = \overline{C}$. Hence, A is a P −cancellation ideal of R.

(⇐) Suppose that A is a P −cancellation ideal of R and to show AM is a P −cancellation module, let $\overline{B}$ and $\overline{C}$ be any two ideals of R_P such that $\overline{B}(AM)_P = \overline{C}(AM)_P$. Then $(\overline{B}A_P)M_P = \overline{B}(A_PM_P) = \overline{B}(AM)_P = \overline{C}(AM)_P = \overline{C}(A_PM_P) = (\overline{C}A_P)M_P$. Now, as M is a P −cancellation module, we get $\overline{B}A_P = \overline{C}A_P$ and as A is a P −cancellation module, we get $\overline{B} = \overline{C}$. Hence, AM is a P −cancellation module.

Next, we give a partial answer to the question that asks, when a P −cancellation module is a cancellation?

Proposition 4.3. Let M be an R −module and P be a prime ideal of R such that $S_R(A) \subseteq P$ for every ideal A of R. If M is a P −cancellation module, then M is a cancellation module.

Proof. To show that M is a cancellation module, let A and B be any two ideals of R such that $AM = BM$, then we have $A_PM_P = (AM)_P = (BM)_P = B_PM_P$, and as M is P −cancellation module, we get that $A_P = B_P$. Now, let $a \in A$, then $\frac{a}{1} \in A_P$ but $A_P = B_P$ so that $\frac{a}{1} \in B_P$, then there exists $u \notin P$ such that $ua \in B$. Now, if possible, suppose that $a \notin B$, then we get $u \in S_R(B)$ and as $S_R(B) \subseteq P$, we get $u \in P$, which is a contradiction. Thus $a \in B$ and hence $A \subseteq B$, similarly we can show $B \subseteq A$. Hence $A = B$ and this shows that M is a cancellation module.

Now, we give the answer to the question that asks, when a cancellation is a P −cancellation module?

Theorem 4.4. Let M be an R −module and P be a prime ideal of R. If $S_M(K) \subseteq P$, for every submodule K of M, then each cancellation submodule of M is a P −cancellation submodule.

Proof. Let N be any cancellation submodule of M. To show that N is a P −cancellation R −module, that is, to show N_P is a cancellation R_P −module. Let $\overline{A}$ and $\overline{B}$ be any two ideals of R_P such that $\overline{A}N_P = \overline{B}N_P$, then $\overline{A} = A_P$ and $\overline{B} = B_P$, for some ideals A and B of R, so that $(AN)_P = A_PN_P = \overline{A}N_P = \overline{B}N_P = B_PN_P = (BN)_P$. As AN and BN are submodules of M so, by the given condition, we have $S_M(AN) \subseteq P$ and $S_M(BN) \subseteq P$ and hence from the last equation, we get $AN = BN$ and as N is a cancellation submodule, we get $A = B$ and hence, we get $\overline{A} = A_P = B_P = \overline{B}$, so that N is a P −cancellation R −module.

As a corollary to the previous theorem, we give the following result.

Corollary 4.5. Let M be an R −module and P be a prime ideal of R such that $S_M(N) \subseteq P$, for every submodule N of M. If M is a cancellation module, then M is a P −cancellation module.

Proof. Since $M \subseteq M$, so the proof of this corollary follows directly by putting $N = M$ in Theorem 4.4.

Now, by combining Proposition 4.3 and Corollary 4.5, we get the following theorem.

Theorem 4.6. Let M be an R −module and P be a prime ideal of R such that $S_R(A) \subseteq P$ for every ideal A of R and $S_M(N) \subseteq P$, for each submodule N of M, then M is a cancellation module if and only if M is a P −cancellation module.

Proof. The proof is clear by Proposition 4.3 and Corollary 4.5.

Proposition 4.7. If M is a finitely generated R −module and P is a prime ideal of R. Then M is a cancellation module if and only if M is a P −cancellation module.

Proof. The proof is analogous to the proof of [11, Proposition 2.3] by taking $S = R \backslash P$.

Finally, we determine some characterizations of cancellation and P −cancellation modules.

Theorem 4.8. If M is an R −module and P is a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for each ideal I of R and each submodule N of M, then the following are equivalent.

(1) M is cancellation module.

(2) M is a P −cancellation module.

(3) If $AM \subseteq BM$, where A and B are ideals of R, then $A \subseteq B$.

(4) If $(a)M \subseteq BM$, where $a \in R$ and B is an ideal of R, then $a \in B$.

(5) $(AM : M) = A$, for all ideal A of R.

(6) $(AM : BM) = (A : B)$, for all ideals A and B of R.

Proof. The proof follows directly from Corollary 4.5, Theorem 3.3, and [10, Theorem 1.9].

References

[1] M. M. Ali and D. J. Smith. *"Pure submodules of Multiplication Modules*". Contributions to Algebra and Geometry. Vol. 45, No. 1, pp. 61-74. (2004).

[2] D. D. Anderson and D. F. Anderson. *"Some remarks on cancellation ideals*". Mathematica Japonica. Vol. 29, No. 6, pp. 879-886. (1984).

[3] S. E. Atani and A. Y. Darani. *"Notes on the Primal Submodules*". Chiang Mai Journal of Science. Vol. 35, No. 3, pp. 399-410. (2008).

[4] D. M. Burton. *"A First Course in Rings and Ideals*". Addison- Wesely Publishing Company, Canada, USA. (1970).

[5] R. Gilmer. *"Multiplicative Ideal Theory".* Marcel Dekker Inc, New York, USA. (1972).

[6] A. K. Jabbar. *"On locally Artinian modules*". International Journal of Algebra. Vol. 6, No. 27. pp. 1325-1334 (2012).

[7] A. K. Jabbar. *"A generalization of prime and weakly prime submodules*". Pure Mathematical Sciences. Vol. 2, No. 1, pp. 1-11. (2013).

[8] C. U. Jensen. *"Arithmetical Rings".* Acta Mathematica Academiae Scientiarum Hungaricae. Vol. 17, No. 1-2, pp. 115-123. (1966).

[9] D. M. Larsen and P. J. McCarthy. *"Multiplicative Theory of Ideals*". Academic Press, New York and London. (1971).

[10] A. S. Mijbass. *"On Cancellation Module*". M.Sc. Thesis, University of Baghdad. (1992).

[11] A. G. Naoum and A. S. Mijbass. *"Weak Cancellation Modules*". Kyungpook Math. J. Vol. 37, pp. 73-82. (1997).

[12] B. N. Shihab. *"On Restricted Cancellation Module*". M.Sc. Thesis, University of Tikrit. (2000).

[13] R. Wisbauer. *"Foundations of Module and Ring Theory*". Gordon and Breach Science Publishers, Germany, (1991).